More practice with definitions, proofs and examples

We will look at some of these questions in the associated optional G12MAN examples classes. Technology permitting, annotated slides and screencasts from those examples classes will be made available soon after each class.

The material covered in these classes is **NEB** (Not Examinable as Bookwork) **except** in as much as it reinforces/repeats bookwork material from lectures. However you may find these sessions helpful as they will provide many more examples of the kind of approach you need when you have to work with definitions in order to prove results or find examples with desired combinations of properties.

You should not hand in your answers to this sheet, and typed solutions will not be available. However, feel free to ask Dr Feinstein (at any time) or one of the G12MAN Workshop Helpers if you wish to discuss your attempted solutions to these questions.

Some of these questions have no direct connection with the material in the G12MAN lecture notes. However some of the questions are similar to (or the same as) some sections of past G12MAN examination questions.

Where we discuss matrices below, you may assume that these matrices have entries which are real numbers if you wish (though the results are also valid when the entries are complex numbers). **We will only discuss square matrices.**

Very quick proofs using definitions

The first few proofs follow very quickly from the basic definitions. You should practise on these until you are fluent (and possibly bored!) with such proofs.

1. Prove that, for every pair of odd integers m and n, we have that their sum $m+n$ is even, and that their product mn is odd.

2. (Revision of matrix definitions.)

(i) Prove that the only square matrices which are **both** upper triangular **and** lower triangular are the diagonal matrices.

(ii) Prove that the only square matrices which are **both** lower triangular **and also** strictly upper triangular are the square zero matrices.

3. Prove that the constant function 0 (the **zero function**) from $\mathbb{R}$ to $\mathbb{R}$ is the only function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is **both** an odd function **and** an even function.

4. Prove that the constant functions from $\mathbb{R}$ to $\mathbb{R}$ are the only functions from $\mathbb{R}$ to $\mathbb{R}$ which are **both** monotone increasing **and** monotone decreasing.

5. Working in $\mathbb{R}$, give a careful proof that 1 is not an interior point of the set $[0,1]$.

6. Prove directly from the definition of interior given in the notes that every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$.

7. Prove that, whenever A and B are bounded subsets of $\mathbb{R}^d$, then $A \cap B$ and $A \cup B$ are also bounded.

8. Let A and B be subsets of $\mathbb{R}$, and consider the **Cartesian product** $A \times B \subseteq \mathbb{R}^2$. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ **or** $B = \emptyset$.

9. Let A, B, C and D be subsets of $\mathbb{R}$. Prove that

$$(A \setminus C) \times (B \setminus D) \subseteq (A \times B) \setminus (C \times D)\,.$$

Quick proofs concerning sets, sequences and absorption

The following **non-standard terminology** will be discussed in lectures in the more general setting of $\mathbb{R}^d$, and (in Chapter 9) for sequences of functions. The results which follow are also true (with appropriate notational changes) in the more general settings.

Note here that when we write 'for all $n \geq N$', by convention we mean 'for all **integers** $n \geq N$'.

Let $(x_n) \subseteq \mathbb{R}$ and let $A \subseteq \mathbb{R}$. We say that the set A **absorbs** the sequence (x_n) if there exists $N \in \mathbb{N}$ such that the following condition holds:

$$\text{for all } n \geq N, \text{ we have } x_n \in A, \qquad (*)$$

i.e., all terms of the sequence from x_N onwards lie in the set A.

If N satisfies condition $(*)$ above, then we also say that A **absorbs the sequence** (x_n) **by stage** N. Recall also that we say that the sequence (x_n) **converges** to a real number x if (and only if) the following condition holds:
For all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|x_n - x| < \varepsilon$.

10. Let $A \subseteq \mathbb{R}$ and let (x_n) be a sequence of real numbers. Let N and N' be positive inetegers with $N \leq N'$.

Suppose that A absorbs the sequence (x_n) by stage N. Show that A also absorbs (x_n) by stage N'.

11. Let $A \subseteq \mathbb{R}$ and let (x_n) be a sequence of real numbers. Show that A absorbs the sequence (x_n) if and only if the set $\{n \in \mathbb{N} \mid x_n \in A^c\}$ is a finite set.

12. Let $B \subseteq \mathbb{R}$ Show that the set $\{n \in \mathbb{N} \mid x_n \in B\}$ is an infinite set if and only if the set B^c does **not** absorb the sequence (x_n).

13. Let A and B be **disjoint** subsets of $\mathbb{R}$ (i.e., $A \cap B = \emptyset$), and let (x_n) be a sequence of real numbers. Prove that it is impossible for **both** A and B to absorb the sequence (x_n).

14. Let A and B be subsets of $\mathbb{R}$ with $A \subseteq B$, and let (x_n) be a sequence of real numbers. Suppose that A absorbs the sequence (x_n). Show that B also absorbs the sequence (x_n).

15. Let A and B be subsets of $\mathbb{R}$. Suppose that every sequence of real numbers (x_n) which is absorbed by A is also absorbed by B. Show that $A \subseteq B$.

16. Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. Prove that the following statements are equivalent (i.e. that each implies the other).

(a) The sequence (x_n) converges to x.

(b) For all $r > 0$, the open interval $]x - r, x + r[$ absorbs the sequence (x_n).

17. Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. Prove that the following statements are equivalent (i.e. that each implies the other).

(a) The sequence (x_n) converges to x.

(b) For all $\delta > 0$, the **closed** interval $[x - \delta, x + \delta]$ absorbs the sequence (x_n).

More practice

The next few proofs still follow fairly quickly from the definitions, but may need a little more work and/or thought. You are also asked to find some examples and do some calculations. (Note that examples also require justification.)

We use the following standard definitions concerning square matrices.
Definition: Let A be a square matrix. Then we say that A is **idempotent** if $A^2 = A$; the matrix A is **nilpotent** if there exists a natural number n such that $A^n = 0$ (the square zero matrix of the appropriate size).
The **trace** of the matrix A is the sum of the entries on the leading (main) diagonal of A.
The **characteristic polynomial** of A, χ_A, is defined by

$$\chi_A(t) = \det(tI - A)$$

(note that some authors use $\det(A - tI)$), where I is the identity matrix of the appropriate size.

18. Let $z \in [0, 10)$. Prove that there exist $x \in [0, 3)$ and $y \in [0, 7)$ with $x + y = z$.

19. Prove that the only square matrices which are both idempotent and nilpotent are the square zero matrices.

20. Give an example of a non-zero 2×2 matrix which is nilpotent.

21. Let A be a (square) diagonal matrix.

(i) Prove that A is idempotent if and only if all of the entries on the diagonal are either 0 or 1.

(ii) Prove that A is nilpotent if and only if $A = 0$.

[Don't forget what you found out above! You MUST use the fact that A is a diagonal matrix here.]

22. Let A be a nilpotent square matrix. Prove that A can not have any non-zero eigenvalues.

23. Let A be an idempotent square matrix and suppose that λ is an eigenvalue for A. Prove that $\lambda \in \{0, 1\}$.

24. **Throughout this question**, let A be a 2×2 square matrix.

(i) By direct calculation, determine formulae for the coefficients of the characteristic polynomial $\chi_A(t)$ in terms of the entries of A.

(ii) What is the relationship between the characteristic polynomial of A, the trace of A and the determinant of A?

(iii) By direct calculation, verify the Cayley-Hamilton Theorem for A.

25. Give an example of a sequence of **finite** subsets A_n of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is **not** a finite subset of $\mathbb{R}$.

26. Give an example of a sequence of **bounded** subsets A_n of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is **not** a bounded subset of $\mathbb{R}$.

27. Give an example of a sequence of **closed** subsets A_n of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is **not** a closed subset of $\mathbb{R}$.

28. Let $(x_n) \subseteq \mathbb{R}$. Suppose that (y_n) is a subsequence of (x_n), and that (z_n) is a subsequence of (y_n). Prove that (z_n) is also a subsequence of (x_n).

[One way to express this result is to say that **every sub-subsequence of a sequence is also a subsequence of that sequence**.]

Even more practice

The remaining questions are of variable difficulty, perhaps depending on the method you choose or whether you spot a helpful idea. You may wish to use some of the results mentioned above to help.

Don't forget that you can also get plenty of practice with proofs and examples by doing the standard question sheet questions, and perhaps trying the questions on the **Challenging problems** sheet.

Sums of subsets of $\mathbb{R}$.

Let A and B be subsets of $\mathbb{R}$, then we define $A + B \subseteq \mathbb{R}$ by

$$A + B = \{x + y \mid x \in A \text{ and } y \in B\} = \{z \in \mathbb{R} \mid \text{there are } x \in A \text{ and } y \in B \text{ with } x + y = z\}$$

i.e., $A + B$ is the set of all real numbers of the form $x + y$ where $x \in A$ and $y \in B$.

29. Prove that, for all integers n, $n^3 - n$ is divisible by three.

30. (i) Prove that there are no positive integers n for which $1 + n + n^2 + n^3$ is prime.

(ii) For which integers n, if any, is $-(1 + n + n^2 + n^3)$ prime?

31. Prove that the only invertible, square matrices which are idempotent are the square identity matrices.

32. Let A be a singular 2×2 matrix.

(i) Prove that, if the trace of A is 1, then A is idempotent.

(ii) Is the converse to the statement in (i) true?

33. Determine **all** 2×2 idempotent matrices.

34. Prove that the only nilpotent, **diagonalizable** square matrices are the square zero matrices.

35. Are there any invertible, nilpotent square matrices?

36. Prove that, for every pair of real numbers x and y with $x \neq y$, there are open intervals U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

37. Prove that limits of convergent sequences are unique: no sequence of real numbers (x_n) can converge to two different real numbers.

[Note here that you must **not** assume that $\lim_{n\to\infty} x_n$ is unique in your proof!]

38. Prove the following set equality involving **sums of sets**, as defined above:

$$[0, 10) = [0, 3) + [0, 7)\,.$$

39. Let R and S be positive real numbers. Prove that

$$[0, R] + [0, S] = [0, R + S]$$

and that

$$[0, R) + [0, S) = [0, R + S)\,.$$

40. Give an example of a pair of subsets A and B of $\mathbb{R}$ such that $A + B \neq A \cup B$.

41. Give an example of a pair of subsets A and B of $\mathbb{R}$ such that $A + B = A \cup B$.

42. Find infinitely many different pairs of subsets A and B of $\mathbb{R}$ such that $A + B = A \cup B$.

43. Which of the three sets $\mathbb{Q}$, $\mathbb{Q}^c$ or $\mathbb{R}$ is $\mathbb{Q} + \mathbb{Q}^c$ equal to?

44. Give examples, with brief justification, of each of the following:

(a) Two unbounded subsets A and B of $\mathbb{R}$ such that $A \cap B$ is bounded;

(b) Two subsets A and B of $\mathbb{R}$ such that neither A nor B is open, but $A \cup B$ is open.

45. Write down, without justification, examples of the following:

(a) A continuous function f from $\mathbb{R}$ to $\mathbb{R}$ and an open subset U of $\mathbb{R}$ such that $f(U)$ is not open in $\mathbb{R}$.

(b) A continuous function f from $\mathbb{R}$ to $\mathbb{R}$ and a closed subset E of $\mathbb{R}$ such that $f(E)$ is not closed in $\mathbb{R}$.

(c) A non-empty, bounded subset E of $\mathbb{R}$ and a continuous function f from E to $\mathbb{R}$ such that $f(E)$ is unbounded.

46. Let (x_n) be a strictly decreasing sequence of positive real numbers. Is it necessarily true that $x_n \to 0$ as $n \to \infty$? [Give a proof or a counterexample.]

47. Let (x_n) be a sequence of real numbers and, for each $n \in \mathbb{N}$, set $y_n = x_n^2$. Suppose that the sequence (y_n) converges. Is it necessarily true that the sequence (x_n) converges? [Give a proof or a counterexample.]

48. Let (x_n) be a sequence of real numbers and, for each $n \in \mathbb{N}$, set $z_n = x_n^3$. Suppose that the sequence (z_n) converges. Is it necessarily true that the sequence (x_n) converges? [Give a proof or a counterexample.]

49. Let (y_n) be a sequence of real numbers. Write down **formal** definitions for what it means to say that: (i) $y_n \to \infty$ as $n \to \infty$; (ii) $y_n \to -\infty$ as $n \to \infty$.

[In this module we treat ∞ and $+\infty$ as the same thing.]

50. Let (x_n) be a sequence of real numbers. Prove that the following statements are equivalent:

(a) $|x_n| \to \infty$ as $n \to \infty$;

(b) (x_n) has no bounded subsequences;

(c) (x_n) has no convergent subsequences.

51. Let E be a non-empty subset of $\mathbb{R}$. Prove that the following statements are equivalent.

(a) The set E is bounded.

(b) Every sequence in E has at least one subsequence which converges in $\mathbb{R}$.

Note in (b) that we do not require the limit of the subsequence to be in E.

52. Let E be a non-empty subset of $\mathbb{R}$. Prove that the following statements are equivalent.

(a) Every sequence in E converges in $\mathbb{R}$.

(b) Every sequence in E converges to a point of E.

(c) The set E has exactly one element.

53. Write down, without justification, examples of the following:

(a) a closed subset E of $\mathbb{R}$ which is not bounded;

(b) a bounded subset E of $\mathbb{R}$ which is not closed;

54. Write down, with brief justification, examples of the following:

(a) a closed subset F of $\mathbb{R}$ which is not sequentially compact;

(b) a bounded subset F of $\mathbb{R}$ which is not sequentially compact;

(c) a sequentially compact subset E of $\mathbb{R}$ such that E is also open;

(d) a sequence in $\mathbb{R}$ which has no convergent subsequences.

55. Give, with brief justification, an example of a non-empty, bounded subset E of $\mathbb{R}$ and a continuous function f from E to $\mathbb{R}$ such that $f(E)$ is unbounded.

56. Give, with brief justification, an example of a divergent sequence of real numbers (x_n) and a non-constant, continuous function f from $\mathbb{R}$ to $\mathbb{R}$, such that the sequence $(f(x_n))_{n=1}^{\infty}$ converges in $\mathbb{R}$.

57. Let x and y be real numbers and let $r > 0$.

(a) Show that there is a real number z such that $|z - x| = r$ and $|z - y| = |x - y| + r$.

(b) Show that the real number z satisfying the condition from part (a) is unique if and only if $x \neq y$.

58. In this question we work with the **Euclidean norm** $\|\cdot\|$ on $\mathbb{R}^d$, where $d \geq 2$ is an integer.

Let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^d$, and let $r > 0$.

(a) Show that there is a $\mathbf{z} \in \mathbb{R}^d$ such that $\|\mathbf{z} - \mathbf{x}\| = r$ and $\|\mathbf{z} - \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\| + r$.

(b) Show that the $\mathbf{z} \in \mathbb{R}^d$ satisfying the condition from part (a) is unique if and only if $\mathbf{x} \neq \mathbf{y}$.